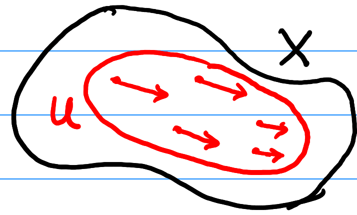
15. Differentials

Big picture

- X scheme $\rightsquigarrow$ $\mathcal{F}$ q.coh. sheaf satisfies $\mathcal{F}|_{\operatorname{Spec} R} = \widetilde{M}$, $\operatorname{Spec} R \subseteq X$ open
- Below: connect local property of varieties (smoothness) & local property of q.coh. sheaves (loc. freeness)
- <u>Idea</u> X smooth variety
  tangent field on $U \subseteq X$ open
  $\triangleq$ nice collection $(v_p \in T_pX)_{p \in U}$
  of tangent vectors



$\rightsquigarrow$ $\mathcal{T}_X$: tangent sheaf $\rightsquigarrow$ loc. free of rank $\dim X$.

<u>Strategy</u> define cotangent sheaf Ω_X for all varieties X/K
$\rightsquigarrow$ $\mathcal{T}_X = \Omega_X^\vee$ dual sheaf.

Ω_X q.coh. $\rightsquigarrow$ first understand it for $X = \operatorname{Spec} R$

<u>Def</u> (Differentials)
Let R be a K-algebra.

$$\Omega_R := \underbrace{\bigoplus_{f \in R} R \cdot \underbrace{df}_{\text{symbol}}}_{\text{generators}} \Big/ \underbrace{\left\langle \begin{array}{ll} d(f+g) = df + dg & , f,g \in R \\ d(f \cdot g) = f dg + g df & , f,g \in R \\ df \;\; = 0 & , f \in K \end{array} \right\rangle}_{\text{relations}}$$

$\rightsquigarrow$ R-module of <u>(Kähler) differentials</u> of R (over K)

$\rightsquigarrow$ $d : R \longrightarrow \Omega_R$ $\quad$ K-linear map (<u>not</u> R-linear!)
$\quad f \longmapsto df$

<u>Think</u> f function $\Rightarrow$ df differential form, $df|_p \in (T_pX)^*$
such that for $v \in T_pX$: $\langle df|_p, v \rangle =$ derivative of f along v

Rmk (Differentials and localization)
$S \subseteq R$ mult. closed subset in K-algebra R

$\rightsquigarrow 0 = d(\frac{1}{f}\cdot f) = \frac{1}{f}df + f d\frac{1}{f} \Rightarrow d(\frac{1}{f}) = -\frac{1}{f^2}df$, $d(\frac{g}{f}) = \frac{fdg - gdf}{f^2}$
in $\Omega_{R[S^{-1}]}$ for $g \in R, f \in S$

$\rightsquigarrow \underbrace{\Omega_{R[S^{-1}]} \xrightarrow{\sim} S^{-1}\Omega_R}_{\text{forming the module of differentials commutes with localization}}$, $d(\frac{g}{f}) \mapsto \frac{1}{f}dg - \frac{g}{f^2}df$

Claim $\exists$ map of sheaves $\mathcal{O}_{\operatorname{Spec} R} \xrightarrow{d} \widetilde{\Omega_R}$ (not map of $\mathcal{O}_X$-mod.)
Pf
$\varphi \in \mathcal{O}_{\operatorname{Spec} R}(U) \rightsquigarrow (\varphi_p \in R_p)_{p\in U} \rightsquigarrow (d\varphi_p \in \Omega_{R_p} = (\Omega_R)_p)_{p \in U} \in \widetilde{\Omega_R}(U)$

$\varphi_p = \frac{g}{f}$ locally $\qquad d\varphi_p = \frac{fdg - gdf}{f^2}$ locally

#

Exa
(a) $R = K[x_1, \dots, x_n] \rightsquigarrow$ for $f \in R$ we have $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\cdot dx_i$ (*)
Exa $d(\underbrace{2x_1 + x_2^2}_{f}) = d(2x_1) + d(x_2 \cdot x_2) = 2\,dx_1 + x_2\cdot dx_2 + x_2 \cdot dx_2$
$= \underbrace{2}_{=\frac{\partial f}{\partial x_1}}\cdot dx_1 + \underbrace{(2x_2)}_{=\frac{\partial f}{\partial x_2}}\cdot dx_2$

No further relations $\rightsquigarrow \Omega_R = R\,dx_1 \oplus \dots \oplus R\,dx_n$
(relations (*) imply all defining relations of Ω_R) free R-module of rank n

(b) $R = A(X) = R[x_1, \dots, x_n]/I(X) \overset{(*)}{\rightsquigarrow} \Omega_R$ generated by dx_i
Additionally: $df = 0$ $\forall f \in I(X)$, since $f = 0 \in A(X) = R$

$\rightsquigarrow \Omega_R = R\,dx_1 \oplus \dots \oplus R\,dx_n / \langle \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} dx_j : i = 1, \dots, m \rangle$ for $I(X) = \langle f_1, \dots, f_m \rangle$
$p \trianglelefteq R$ maximal ($\hat{=}$ point in X)
$\widetilde{\Omega_R}|_p = \Omega_R \otimes_R \underbrace{R/p}_{K} = K\,dx_1 \oplus \dots \oplus K\,dx_n / \langle \sum \frac{\partial f_i}{\partial x_j}(p)\, dx_j : i \rangle \underset{\text{[Pro 10.11]}}{\cong} (T_p X)^\vee$

The cotangent sheaf

Have seen K-algebra $R \rightsquigarrow \Omega_R$ module of differentials
$\rightsquigarrow$ get sheaf $\widetilde{\Omega}_R$ on $\operatorname{Spec} R$

Problem for X any variety: don't want to glue these by hand
Solution Find global construction which gives
$\widetilde{\Omega}_R$ on $\operatorname{Spec} R$.

Lem (Alternative construction of Ω_R)
Let R be a K-algebra. Consider the map

$$\delta: R \otimes_K R \longrightarrow R, \quad f \otimes g \mapsto fg$$

and set $J := \operatorname{Ker}(\delta)$. Then $J/J^2 \cong \Omega_R$ as R-modules.

Pf What R-module structure?
$\rightsquigarrow R \otimes_K R$ is R-module via $a \cdot (b \otimes c) = (ab) \otimes c$ or
$a \cdot (b \otimes c) = b \otimes ac$. } different! ($\otimes_K$)

What is J^2?

$$J^2 = \left\langle \sum_{i,j} f_{1,i} \cdot f_{2j} \otimes g_{1,i} \cdot g_{2j} \ : \ \sum_{i=1}^n f_{1,i} \otimes g_{1,i}, \ \sum_{j=1}^m f_{2,j} \otimes g_{2,j} \in J \right\rangle$$

Submodule with respect to either R-module structure above

$\rightsquigarrow$ can form J/J^2 (as abelian group), two potentially different R-module structures

Claim these structures are the same!
Pf $h \in R$, $\varphi = \sum_{i=1}^n f_i \otimes g_i \in J$

$$\rightsquigarrow \sum_{i=1}^n (\underbrace{f_i \otimes h g_i}_{h \cdot \varphi} - \underbrace{h f_i \otimes g_i}_{h \cdot \varphi}) = \underbrace{\left(\sum_{i=1}^n f_i \otimes g_i\right)}_{\in J} \cdot \underbrace{(1 \otimes h - h \otimes 1)}_{\in J} \in J^2$$

two different R-mod. struct.

$\rightsquigarrow$ get canonical R-module structure on J/J^2

<u>Check</u> $\quad J/J^2 \rightleftarrows \Omega_R \quad$ are inverse R-module homomorphisms

$$[\textstyle\sum_i f_i \otimes g_i] \longmapsto f_i\, dg_i$$
$$[1\otimes f - f\otimes 1] \longleftarrow\!\shortmid\; df$$

$\square$

<u>Construction (Cotangent sheaf)</u>

X variety $\Rightarrow \Delta_X \subseteq X\times X$ closed subvariety, $i = (\mathrm{id}_X, \mathrm{id}_X) : X \longrightarrow X\times X$ closed subscheme

$\mathcal{J} := \mathcal{J}_{X/X\times X}$ ideal sheaf on $X\times X$

$X = \operatorname{Spec} R$: morphism $i \mathrel{\hat{=}} \delta : R\otimes_K R \to R$ from above

(cf. video 12.14 $\delta(f\otimes g) = \mathrm{id}_R(f)\cdot \mathrm{id}_R(g)$)

$0 \longrightarrow J \longrightarrow R\otimes_K R \longrightarrow R \longrightarrow 0$ exact seq. of $R\otimes_K R$-modules

$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_{X\times X} \longrightarrow i_*\mathcal{O}_X \longrightarrow 0$ exact seq. of $\mathcal{O}_{X\times X}$-modules

and $\mathcal{O}_{X\times X} = \widetilde{R\otimes_K R}$, $i_*\mathcal{O}_X = \widetilde{R}$

$\Rightarrow \mathcal{J} = \widetilde{J}$ for $J = \ker(\delta)$

$\Rightarrow \mathcal{J}/\mathcal{J}^2 = \widetilde{J/J^2}$ on $X\times X$ $\Rightarrow i^*\mathcal{J}/\mathcal{J}^2 = \widetilde{\underline{J/J^2}}$ on X

see as R-module now $\mathrel{\hat{=}} \Omega_R$

X any variety

$\Rightarrow \Omega_X := i^*\mathcal{J}/\mathcal{J}^2 \quad$ <u>cotangent sheaf</u> of X

restricts to $\widetilde{\Omega_R}$ on $\operatorname{Spec} R \subseteq X$ open

$\rightsquigarrow d : \mathcal{O}_X \longrightarrow \Omega_X \quad$ map of sheaves (<u>not</u> morph. of $\mathcal{O}_X$-mod.)

compute on affine patches

A criterion for smoothness and the tangent bundle

X smooth variety, pure dimension n

$\Rightarrow$ all spaces T_pX (and thus $T_pX^\vee$) have dim n $\forall p \in X$

& $\Omega_X|_p = T_pX^\vee$ ([Exa. 15.3 b)]: checked on $\operatorname{Spec} R \subseteq X$)

<u>Expect</u>: Ω_X loc. free of rank n

<u>Pro</u> X variety of pure dimension n. Then:

Ω_X loc. free of rank n $\Leftrightarrow$ X is smooth.

<u>Pf</u> "$\Rightarrow$" Ω_X vect. bundle of rk. n $\Rightarrow$ $\Omega_X|_p = T_pX^\vee$ dim n $\forall p \in X$

$\Rightarrow$ dim $T_pX = n$ $\forall p \in X$ $\xRightarrow{[\text{Lem } 10.9]}$ X is smooth (at all $p \in X$).

"$\Leftarrow$" X smooth, $p \in X$

being locally free is local property $\rightsquigarrow$ assume $X \subseteq \mathbb{A}^r$ affine

$$R = A(X) = K[x_1, \dots, x_r] / \langle f_1, \dots, f_m \rangle$$

$$\overset{[\text{Exa } 15.3 b]}{\rightsquigarrow} (T_pX)^\vee = K dx_1 \oplus \dots \oplus K dx_r / \left\langle \sum_{j=1}^r \frac{\partial f_i}{\partial x_j}(p)\, dx_j \; : i = 1, \dots, m \right\rangle$$

$(T_pX)^\vee$ dim n $\rightsquigarrow$ $J(p) = \left(\frac{\partial f_i}{\partial x_j}(p)\right)_{i,j}$ has rank $r-n$

<u>Assume</u> submatrix of last $r-n$ rows & columns has nonzero determinant $h \in R$ at p.

$\rightsquigarrow$ $dx_{n+1}, \dots, dx_r$ can be expressed in basis $dx_1, \dots, dx_n$ of $(T_pX)^\vee$

On $D(h) \subseteq X$: invert determinant, express $dx_{n+1}, \dots, dx_r$ as R_h-lin. combination of $dx_1, \dots, dx_n$

$\Rightarrow$ $dx_1, \dots, dx_n$ generate Ω_{R_h} as R_h-module

<u>Claim</u> no relations $\rightsquigarrow \Omega_{R_h} = R_h\cdot dx_1 \oplus \cdots \oplus R_h \cdot dx_n$ is free

<u>Pf</u> Assume $g_1\cdot dx_1 + \cdots + g_n\, dx_n = 0$ for $g_1,\dots,g_n \in R_h$ non-zero

Say $g_1 \neq 0 \xRightarrow{\text{Nullst.satz}} \exists\, q \in D(h)$ with $g_1(q) \neq 0$

$$\Rightarrow \underbrace{g_1(q)}_{\neq 0}\cdot \underbrace{dx_1} + \cdots + g_n(q)\cdot \underbrace{dx_n} = 0 \in (T_qX)^\vee$$

$dx_1, \dots, dx_n$ generate $(T_qX)^\vee$

$\Rightarrow \dim (T_qX)^\vee < n$ ↯ to assumption. ※

$\rightsquigarrow \Omega_{R_h}$ is free R_h-module of rk n

$\rightsquigarrow \Omega_X$ loc. free of rank n. □

<u>Def</u> (Tangent bundle)

X smooth variety, pure dimension n

Ω_X <u>cotangent bundle</u>

$T_X = \Omega_X^\vee$: <u>tangent sheaf / bundle</u>

<u>Exa</u> $X = \mathbb{P}^1 \rightsquigarrow \Omega_X = \mathcal{O}_{\mathbb{P}^1}(-2)$, $T_X = \mathcal{O}_{\mathbb{P}^1}(2)$

$\Rightarrow \Omega_X(\mathbb{P}^1) = \{0\}$: $\nexists$ nonzero algebraic differential form on $\mathbb{P}^1$

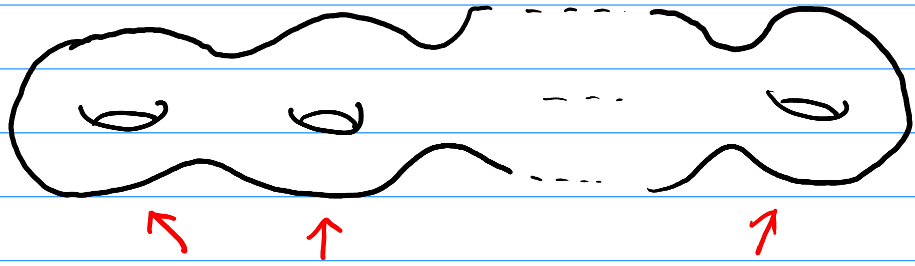
<u>Pf</u> See [Gathmann, Pro. 15.8, Rmk. 15.9]. □

Application: The genus of a smooth projective curve

$K = \mathbb{C}$, X a smooth, projective, connected variety of dimension $1/\mathbb{C}$
Let $S = X(\mathbb{C})$ with complex topology (e.g. $S \subseteq \mathbb{P}^n_{\mathbb{C}}$ w/ rel. top.)

$\rightsquigarrow$ X smooth of dim. 1 $\Rightarrow$ S is complex manifold of dim 1
(open cover U_i with $U_i \cong V_i \overset{\text{open}}{\subseteq} \mathbb{C}$)
$\Rightarrow$ S is oriented real mfld. of dim 2
surface
$\rightsquigarrow$ X projective $\Rightarrow$ S is compact
$\rightsquigarrow$ X connected $\Rightarrow$ S is connected

Thm (Classification thm. of closed surfaces)
Any compact connected oriented surface S is homeomorphic to the connected sum of $g \geq 0$ tori:

$S \underset{\text{homeo.}}{\cong}$ 

g holes

X variety as above $\rightsquigarrow g(X) = g \in \mathbb{N}$ genus of X

Q Can we calculate this purely using algebraic geometry?

Pro/Def X smooth, projective connected curve
$\Rightarrow \Omega_X(X)$ is finite dimensional $K = \mathcal{O}_X(X)$ - vector space

$g_a(X) := \dim_K \Omega_X(X)$ arithmetic genus of X

Thm $g(X) = g_a(X)$.

Using methods from the final chapters of [Gathmann] you can prove:

<u>Pro</u> $X \subseteq \mathbb{P}^2$ smooth curve of degree $d \in \mathbb{N}_{>0}$

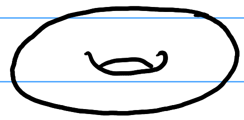
$$\Longrightarrow g_a(X) = \frac{(d-1)(d-2)}{2}$$

<u>Exa</u>

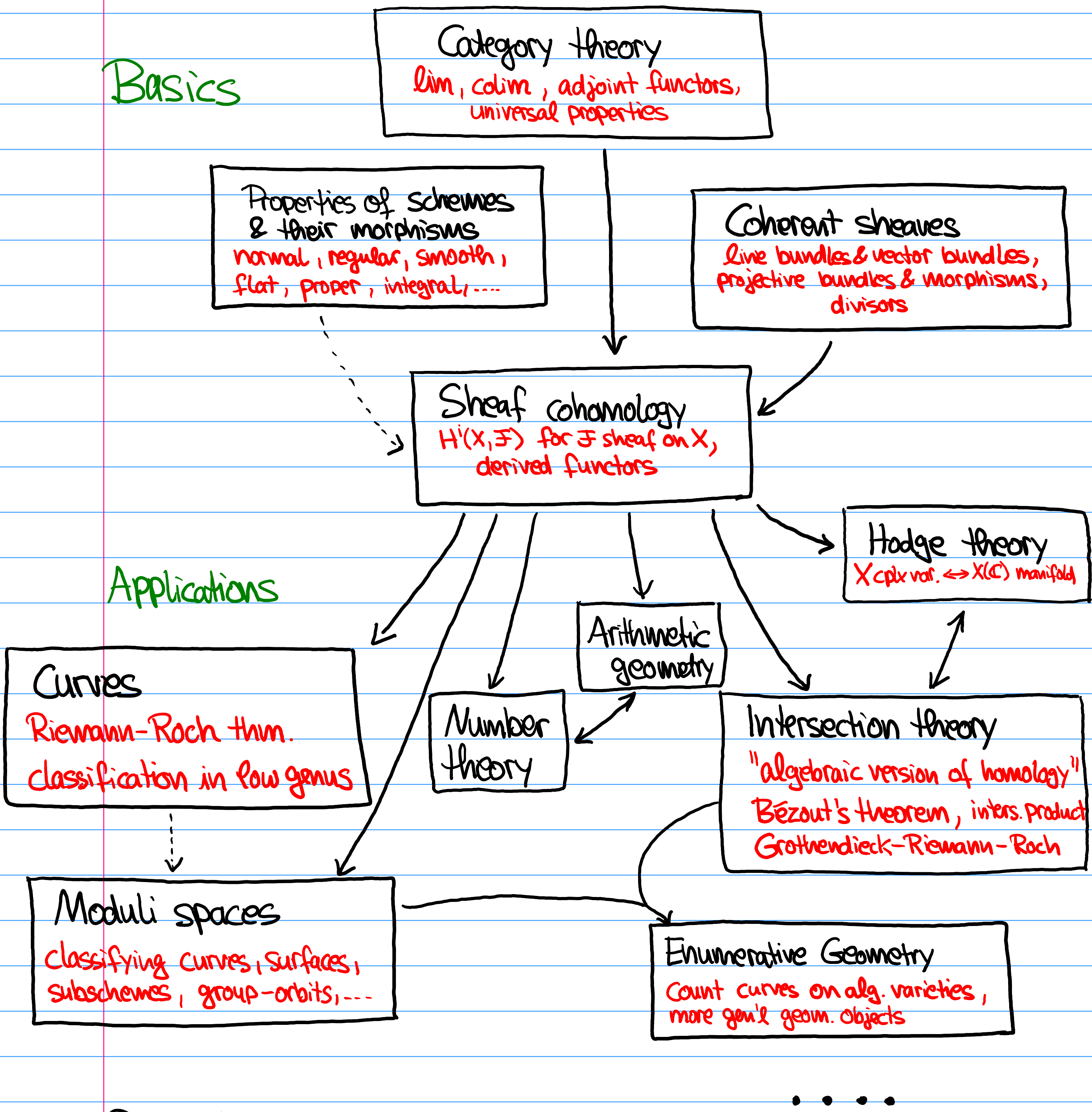
- $d = 1,2$: X = line / conic $\xRightarrow{\text{proved}} X \cong \mathbb{P}^1$

$$g_a(X) = \dim_K \Omega_{\mathbb{P}^1}(\mathbb{P}^1) = \dim_K \underbrace{\mathcal{O}_{\mathbb{P}^1}(-2)(\mathbb{P}^1)}_{=\{0\}} = 0 = \frac{(d-1)(d-2)}{2}$$

- $d = 3$: X = smooth cubic curve ("elliptic curve" – Presence Sheet 6)

$g_a(X) = \frac{(3-1)(3-2)}{2} = 1 \rightsquigarrow$  $X(\mathbb{C}) \underset{\text{homeo}}{\cong} S^1 \times S^1$

Appendix: What next?



Practically speaking:

→ finish [Gathmann]: sheaf cohomology

→ can recommend: [Vakil – The rising sea: Found. of alg. geom.]

⇝ takes more time to study examples, fill details, prove Lemmas, ...